

# CHERN-OSSERMAN TYPE EQUALITY FOR COMPLETE SURFACES IN $\mathbb{R}^N$

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**ABSTRACT.** We obtain a Chern-Osserman type equality of a complete properly immersed surface in Euclidean space, provided the  $L^2$ -norm of the second fundamental form is finite. Also, by using a monotonicity formula, we prove that if the  $L^2$ -norm of mean curvature of a noncompact surface is finite, then it has at least quadratic area growth.

## 1. INTRODUCTION

Let  $M$  be a complete minimal surface in  $\mathbb{R}^n$  with finite total curvature, Chern and Osserman [2], [7] proved that

$$(1.1) \quad -\chi(M) \leq -\frac{1}{2\pi} \int_M K - k,$$

where  $K$  is the Gauss curvature of  $M$ ,  $\chi(M)$  is the Euler characteristic of  $M$  and  $k$  is the number of ends of  $M$ . Further results were obtained by Jorge and Meeks [5] that

$$(1.2) \quad -\chi(M) = -\frac{1}{2\pi} \int_M K - \lim_{t \rightarrow \infty} \frac{\text{area}(M \cap B(t))}{\pi t^2},$$

where  $B(t)$  is the extrinsic ball of radius  $t$ .

When  $M$  is a general surface properly immersed in  $\mathbb{R}^n$  with  $\int_M |A|^2 < \infty$ , where  $A$  is the second fundamental form of the immersion, White [9] proved that  $\frac{1}{2\pi} \int_M K$  must be an integer. In this paper, we present a general version of (1.2), where  $M$  is a general surface properly immersed in  $\mathbb{R}^n$  with the  $L^2$ -norm of the second fundamental form is finite.

**Theorem 1.1.** *Let  $M$  be a complete properly immersed noncompact oriented surface in  $\mathbb{R}^n$ ,  $A$  the second fundamental form of the immersion,  $r$  the distance of  $\mathbb{R}^n$  from a fixed point and  $M_t = \{x \in M : r(x) < t\}$ ,  $\chi(M)$  the Euler characteristic of  $M$ . Suppose  $\int_M |A|^2 < \infty$ , then*

1.  $\lim_{t \rightarrow \infty} \frac{\text{area} M_t}{\pi t^2}$  exists and is a positive integer;
2.  $\lim_{t \rightarrow \infty} \frac{\text{area} M_t}{\pi t^2} = \chi(M) - \frac{1}{2\pi} \int_M K$ .

Since  $\int_M |A|^2 < +\infty$ , then  $\int_M |K| < +\infty$  by Gauss equation. When  $M$  is a complete surface with finite total Gaussian curvature, Huber [4] proved that  $M$  has finite topological

type. And Cohn-Vossen [3] obtained:

$$(1.3) \quad 2\pi\chi(M) - \int_M K \geq 0.$$

The explicit equality was obtained by Shiohama [8]:

$$(1.4) \quad \chi(M) - \frac{1}{2\pi} \int_M K = \lim_{t \rightarrow \infty} \frac{D(t)}{\pi t^2},$$

where  $D(t)$  denote the area of geodesic balls of radius  $t$  at a fixed point. Our theorem shows that (1.4) also holds with extrinsic balls instead of geodesic balls if  $M$  is properly immersed in  $\mathbb{R}^n$ .

The proof of Theorem 1.1 is based on two monotonicity formulas (Theorem 2.4). The monotonicity formulas also have an interesting application, namely, if the  $L^2$ -norm of mean curvature  $H$  of the surface is finite, then it has at least quadratic area growth.

**Corollary 1.2.** (see also Corollary 2.5) *Let  $M$  be a complete properly immersed noncompact surface in  $\mathbb{R}^n$  with  $\int_M |H|^2 < \infty$ , then the volume of the intersection of  $M$  and the extrinsic balls has at least quadratic area growth.*

## 2. PRELIMINARIES

Let  $x : M \rightarrow \mathbb{R}^n$  be a complete properly immersed surface in  $\mathbb{R}^n$ ,  $r$  the distance function of  $\mathbb{R}^n$  from a fixed point. For simplicity, we always assume the fixed point to be 0, unless otherwise specific. Denote the covariant derivative of  $\mathbb{R}^n$  and  $M$  by  $\bar{\nabla}$  and  $\nabla$  respectively. Let  $X, Y$  be two tangent vector fields of  $M$ , then

$$(2.1) \quad \begin{aligned} (\bar{\nabla}^2 r)(X, Y) &= XY(r) - \bar{\nabla}_X Y(r) \\ &= (\nabla^2 r)(X, Y) - \langle A(X, Y), \bar{\nabla} r \rangle. \end{aligned}$$

The equality (2.1), together with the fact that  $\bar{\nabla}^2 r = \frac{1}{r}(g_{st} - dr \otimes dr)$ , where  $g_{st}$  denotes the standard metric of  $\mathbb{R}^n$ , implies

**Proposition 2.1.** *For any unit tangent vector  $e$  of  $M$ ,*

$$(\nabla^2 r)(e, e) = \frac{1}{r}(1 - \langle e, \nabla r \rangle^2) + \langle A(e, e), \nabla^\perp r \rangle,$$

where  $\nabla^\perp r$  is the projection of  $\bar{\nabla} r$  onto the normal of  $M$ .

By Sard's theorem, for a.e.  $t > 0$ ,  $M_t = \{x \in M : r(x) < t\}$  is a related compact open subset of  $M$  with the boundary  $\partial M_t$  being a closed immersed curve of  $M$ . Let  $v(t) = \text{area} M_t$ ,  $A$  the second fundamental form of  $M$ , and  $H = \text{tr} A$  the mean curvature vector.

**Proposition 2.2.** *Suppose  $M$  is a complete properly immersed surface in  $\mathbb{R}^n$ . Then for a.e.  $t > 0$ ,*

$$2\pi\chi(M_t) - \int_{M_t} K = \frac{1}{t} \left( v'(t) + \int_{\partial M_t} \frac{\langle x^\perp, H \rangle}{|\nabla r|} \right) - \int_{\partial M_t} \left\langle A \left( \frac{\nabla r}{|\nabla r|}, \frac{\nabla r}{|\nabla r|} \right), \frac{\nabla^\perp r}{|\nabla r|} \right\rangle,$$

where  $x^\perp$  is the projection of position vector  $x$  onto the normal of  $M$ .

*Proof.* By the Gauss-Bonnet formula, it's sufficient to verify

$$(2.2) \quad \int_{\partial M_t} k_g = \frac{1}{t}(v'(t) + \int_{\partial M_t} \frac{\langle x^\perp, H \rangle}{|\nabla r|}) - \int_{\partial M_t} \langle A(\frac{\nabla r}{|\nabla r|}, \frac{\nabla r}{|\nabla r|}), \frac{\nabla^\perp r}{|\nabla r|} \rangle,$$

where  $k_g$  denote the geodesic curvature of  $\partial M_t$  in  $M$ .

Suppose  $e$  is the unit tangent vector of  $\partial M_t$ . Since the normal of  $\partial M_t$  is  $\frac{\nabla r}{|\nabla r|}$ ,

$$(2.3) \quad \begin{aligned} k_g &= -\langle \nabla_e e, \frac{\nabla r}{|\nabla r|} \rangle \\ &= \frac{1}{|\nabla r|} (\nabla^2 r)(e, e) \\ &= \frac{1}{|\nabla r|} \left( \frac{1}{r} + \langle A(e, e), \nabla^\perp r \rangle \right) \\ &= \frac{1}{|\nabla r|} \left( \frac{1}{r} + \langle H - A(\frac{\nabla r}{|\nabla r|}, \frac{\nabla r}{|\nabla r|}), \nabla^\perp r \rangle \right), \end{aligned}$$

where the third equality follows by Proposition 2.1. Then by using co-area formula,  $v'(t) = \int_{\partial M_t} \frac{1}{|\nabla r|}$  and the fact that  $\nabla^\perp r = \frac{x^\perp}{r}$ , we obtain (2.2).  $\square$

**Proposition 2.3.** *Let  $M$  be a complete properly immersed surface in  $\mathbb{R}^n$ , then*

$$tv'(t) = t \int_{\partial M_t} \frac{|\nabla^\perp r|^2}{|\nabla r|} + 2v(t) + \int_{M_t} \langle x^\perp, H \rangle.$$

*Proof.* Since  $\frac{1}{2}\Delta r^2 = 2 + \langle x, H \rangle$ , integrating over  $M_t$  and using the Green's formula,

$$(2.4) \quad t \int_{\partial M_t} |\nabla r| = 2v(t) + \int_{M_t} \langle x, H \rangle.$$

By the co-area formula ,

$$v'(t) = \int_{\partial M_t} \frac{1}{|\nabla r|}.$$

So we have,

$$\begin{aligned} tv'(t) &= t \left( \int_{\partial M_t} \frac{1}{|\nabla r|} - \int_{\partial M_t} |\nabla r| \right) + t \int_{\partial M_t} |\nabla r| \\ &= t \int_{\partial M_t} \frac{|\nabla^\perp r|^2}{|\nabla r|} + 2v(t) + \int_{M_t} \langle x^\perp, H \rangle. \end{aligned} \quad \square$$

**Theorem 2.4.** *Let  $M$  be a complete properly immersed surface in  $\mathbb{R}^n$ ,  $r$  the distance of  $\mathbb{R}^n$  from a fixed point  $x_0$ ,  $H$  the mean curvature of  $M$ ,  $M_t = \{x \in M : r(x) < t\}$ ,  $v(t) = \text{area} M_t$ , then both*

$$u_1(t) \triangleq \frac{v(t)}{t^2} - \frac{1}{2t^2} \int_{M_t} |(x - x_0)^\perp| |H| + \frac{1}{16} \int_{M_t} |H|^2$$

and

$$u_2(t) \triangleq \frac{v(t)}{t^2} - \frac{1}{t^2} \int_{M_t} |(x - x_0)^\perp| |H| + \frac{1}{4} \int_{M_t} |H|^2$$

are monotone nondecreasing in  $t$ .

*Proof.* For simplicity, we assume  $x_0 = 0$ . By Proposition 2.3, we have

$$(2.5) \quad tv'(t) \geq t \int_{\partial M_t} \frac{|\nabla^\perp r|^2}{|\nabla r|} + 2v(t) - \int_{M_t} |x^\perp| |H|.$$

By co-area formula and the weighted mean value inequalities,

$$(2.6) \quad \begin{aligned} \frac{d}{dt} \left( \int_{M_t} |x^\perp| |H| \right) &= \int_{\partial M_t} \frac{|x^\perp| |H|}{|\nabla r|} \\ &= t \int_{\partial M_t} \frac{|\nabla^\perp r| |H|}{|\nabla r|} \\ &\leq 2 \int_{\partial M_t} \frac{|\nabla^\perp r|^2}{|\nabla r|} + \frac{t^2}{8} \int_{\partial M_t} \frac{|H|^2}{|\nabla r|}. \end{aligned}$$

Combining (2.5) and (2.6), we have

$$(2.7) \quad tv'(t) \geq \frac{t}{2} \left( \left( \int_{M_t} |x^\perp| |H| \right)' - \frac{t^2}{8} \int_{\partial M_t} \frac{|H|^2}{|\nabla r|} \right) + 2v(t) - \int_{M_t} |x^\perp| |H|,$$

or equivalently,

$$(2.8) \quad tv'(t) - 2v(t) - \frac{1}{2} \left( t \left( \int_{M_t} |x^\perp| |H| \right)' - 2 \int_{M_t} |x^\perp| |H| \right) + \frac{t^3}{16} \int_{\partial M_t} \frac{|H|^2}{|\nabla r|} \geq 0.$$

Dividing both sides of (2.8) by  $t^3$  yields

$$(2.9) \quad \frac{d}{dt} \left( \frac{v(t)}{t^2} - \frac{1}{2} \frac{\int_{M_t} |x^\perp| |H|}{t^2} + \frac{1}{16} \int_{M_t} |H|^2 \right) \geq 0,$$

this proves that  $u_1(t)$  is monotone nondecreasing in  $t$ .

If we make slight modifications to (2.5) and (2.6), we have

$$(2.5)' \quad tv'(t) \geq t \int_{\partial M_t} \frac{|\nabla^\perp r|^2}{|\nabla r|} + 2v(t) - 2 \int_{M_t} |x^\perp| |H|,$$

and

$$(2.6)' \quad \frac{d}{dt} \left( \int_{M_t} |x^\perp| |H| \right) \leq \int_{\partial M_t} \frac{|\nabla^\perp r|^2}{|\nabla r|} + \frac{t^2}{4} \int_{\partial M_t} \frac{|H|^2}{|\nabla r|}.$$

Combining (2.5)' and (2.6)', we obtain

$$(2.9)' \quad \frac{d}{dt} \left( \frac{v(t)}{t^2} - \frac{\int_{M_t} |x^\perp| |H|}{t^2} + \frac{1}{4} \int_{M_t} |H|^2 \right) \geq 0,$$

i.e.  $u_2(t)$  is monotone nondecreasing in  $t$ . □

*Remark 2.4.* From the poof, we can see that the theorem is also valid for noncomplete surface, for  $t$  with  $\partial M \cap B_{x_0}(t) = \emptyset$ , where  $B_{x_0}(t)$  is the ball in  $\mathbb{R}^n$  of radius  $t$  and centered at  $x_0$ .

By Theorem 2.4, we can get various volume estimates under suitable restrictions on mean curvature  $H$ .

**Corollary 2.5.** *Let  $M$  be a complete properly immersed noncompact surface in  $\mathbb{R}^n$  with  $\int_M |H|^2 < \infty$ , then the volume of the intersection of  $M$  and the extrinsic balls has at least quadratic area growth.*

*Proof.* Without loss of generality, we assume the center of the extrinsic balls to be 0. Since  $\int_M |H|^2 < \infty$ , for a given  $\varepsilon > 0$ , there exists  $R > 0$ , such that

$$\int_{M \setminus B_0(R)} |H|^2 < \varepsilon.$$

Now for  $t > R$  large enough, choosing a point  $p \in M \cap \partial B_0(\frac{t+R}{2})$ , then  $B_p(\frac{t-R}{2}) \subset B_0(t) \setminus B_0(R)$ , so we have

$$(2.10) \quad \int_{M \cap B_p(\frac{t-R}{2})} |H|^2 < \varepsilon, \quad \text{Vol}(M \cap B_0(t)) \geq \text{Vol}(M \cap B_p(\frac{t-R}{2})).$$

Taking  $x_0 = p$  in Theorem 2.4, then we have

$$(2.11) \quad u_1(\frac{t-R}{2}) \geq \lim_{r \rightarrow 0} u_1(r) = \pi.$$

Combining (2.10) and (2.11), we obtain

$$\begin{aligned} \text{Vol}(M \cap B_0(t)) &\geq \text{Vol}(M \cap B_p(\frac{t-R}{2})) \\ &\geq \frac{(t-R)^2}{4} \left( u_1(\frac{t-R}{2}) - \frac{1}{16} \int_{M_{\frac{t-R}{2}}} |H|^2 \right) \\ &\geq \frac{\pi - \varepsilon}{4} (t-R)^2. \end{aligned}$$

The conclusion follows by choosing  $\varepsilon$  small. □

### 3. PROOF OF THEOREM 1.1

**Lemma 3.1.** *Let  $M$  be as in Theorem 1.1, then both  $\lim_{t \rightarrow \infty} \frac{v(t)}{t^2}$  and  $\lim_{t \rightarrow \infty} \frac{\int_{M_t} |x^\perp| |H|}{t^2}$  exist.*

*Proof.* First we prove:

**Claim:**  $\liminf_{t \rightarrow \infty} \frac{v}{t^2} < +\infty$ .

**Proof of the claim:** Since by the weighted mean value inequality,

$$(3.1) \quad \left| \frac{1}{t} \int_{\partial M_t} \frac{\langle x^\perp, H \rangle}{|\nabla r|} \right| \leq \int_{\partial M_t} \frac{|H|}{|\nabla r|} \leq \frac{1}{2} \left( t \int_{\partial M_t} \frac{|H|^2}{|\nabla r|} + \frac{v'}{t} \right),$$

by Proposition 2.2, we have

$$(3.2) \quad \begin{aligned} 2\pi\chi(M_t) - \int_{M_t} K &\geq \frac{1}{t} \left( v'(t) - \int_{\partial M_t} \frac{\langle x^\perp, H \rangle}{|\nabla r|} \right) - \int_{\partial M_t} |A| \frac{|\nabla^\perp r|}{|\nabla r|} \\ &\geq \frac{v'}{2t} - \frac{t}{2} \int_{\partial M_t} \frac{|H|^2}{|\nabla r|} - \int_{\partial M_t} \left( \frac{t|A|^2}{2|\nabla r|} + \frac{|\nabla^\perp r|^2}{2t|\nabla r|} \right) \\ (by \text{ Proposition 2.3}) &= \frac{v'}{2t} - \frac{t}{2} \int_{\partial M_t} \frac{|H|^2}{|\nabla r|} - \frac{t}{2} \int_{\partial M_t} \frac{|A|^2}{|\nabla r|} - \frac{tv' - (2v(t) + \int_{M_t} \langle x^\perp, H \rangle)}{2t^2} \\ &= \frac{v}{t^2} - \frac{t}{2} \int_{\partial M_t} \frac{|H|^2}{|\nabla r|} - \frac{t}{2} \int_{\partial M_t} \frac{|A|^2}{|\nabla r|} + \frac{\int_{M_t} \langle x^\perp, H \rangle}{2t^2} \\ &\geq \frac{v}{t^2} - \frac{t}{2} \int_{\partial M_t} \frac{|H|^2}{|\nabla r|} - \frac{t}{2} \int_{\partial M_t} \frac{|A|^2}{|\nabla r|} - \frac{\sqrt{v \int_{M_t} |H|^2}}{2t} \\ &\geq \frac{v}{2t^2} - \frac{t}{2} \int_{\partial M_t} \frac{|H|^2}{|\nabla r|} - \frac{t}{2} \int_{\partial M_t} \frac{|A|^2}{|\nabla r|} - \frac{1}{8} \int_{M_t} |H|^2, \end{aligned}$$

where we use the weighted mean value inequalities in the second and the last equality, while the second equality count backwards follows from Cauchy's inequality.

Since  $\int_M |H|^2 + \int_M |A|^2 < +\infty$ , there exists a sequence  $\{\tau_i\}$  diverging to infinity such that

$$(3.3) \quad t \left( \int_{\partial M_t} \frac{|H|^2}{|\nabla r|} + \int_{\partial M_t} \frac{|A|^2}{|\nabla r|} \right) \Big|_{t=\tau_i} \longrightarrow 0 \quad as \quad i \longrightarrow \infty.$$

Otherwise, we must have  $\liminf_{t \rightarrow \infty} t \left( \int_{\partial M_t} \frac{|H|^2}{|\nabla r|} + \int_{\partial M_t} \frac{|A|^2}{|\nabla r|} \right) = \delta > 0$ . So for sufficient large  $t$ , we have

$$t \left( \int_{\partial M_t} \frac{|H|^2}{|\nabla r|} + \int_{\partial M_t} \frac{|A|^2}{|\nabla r|} \right) > \frac{\delta}{2},$$

i.e.

$$\int_{\partial M_t} \frac{|H|^2}{|\nabla r|} + \int_{\partial M_t} \frac{|A|^2}{|\nabla r|} > \frac{\delta}{2t}.$$

When you integrate  $t$ , by the co-area formula, it is as bounded on the left as it is diverging on the right, a contradiction.

Then taking  $t = \tau_i$  in (3.2), together with the fact that

$$\chi(M_t) \leq 1, \quad \left| \int_{M_t} K \right| \leq \frac{1}{2} \int_M |A|^2 < +\infty \quad and \quad \int_{M_t} |H|^2 \leq 2 \int_M |A|^2 < +\infty,$$

we have  $\limsup_{i \rightarrow \infty} \frac{v(t_i)}{t_i^2} < +\infty$ , which implies  $\liminf_{t \rightarrow \infty} \frac{v}{t^2} \leq \limsup_{i \rightarrow \infty} \frac{v(t_i)}{t_i^2} < +\infty$ . This proves the claim.

Let  $u_1(t)$  and  $u_2(t)$  be as in Theorem 2.4 with  $x_0 = 0$ . By the claim, we have

$$(3.4) \quad \begin{aligned} \liminf_{t \rightarrow \infty} u_1(t) &\leq \liminf_{t \rightarrow \infty} \frac{v(t)}{t^2} + \frac{1}{16} \int_M |H|^2 < +\infty, \\ \liminf_{t \rightarrow \infty} u_2(t) &\leq \liminf_{t \rightarrow \infty} \frac{v(t)}{t^2} + \frac{1}{4} \int_M |H|^2 < +\infty. \end{aligned}$$

Combining (3.4) and Theorem 2.4, we know that both  $u_1(t)$  and  $u_2(t)$  have finite limit as  $t \rightarrow \infty$ .

Since

$$(3.5) \quad \begin{aligned} \frac{v(t)}{t^2} &= 2u_1(t) - u_2(t) + \frac{1}{8} \int_{M_t} |H|^2, \\ \frac{\int_{M_t} |x^\perp| |H|}{t^2} &= 2u_1(t) - 2u_2(t) + \frac{3}{8} \int_{M_t} |H|^2, \end{aligned}$$

we conclude that both  $\lim_{t \rightarrow \infty} \frac{v(t)}{t^2}$  and  $\lim_{t \rightarrow \infty} \frac{\int_{M_t} |x^\perp| |H|}{t^2}$  exist.  $\square$

**Lemma 3.2.** *There exists a sequence  $\{t_k\}$  diverging to infinity such that*

$$(i) \quad \lim_{k \rightarrow \infty} \frac{v'(t_k)}{t_k} = \lim_{k \rightarrow \infty} \frac{2v(t_k)}{t_k^2}, \quad \lim_{k \rightarrow \infty} \frac{2}{t_k^2} \int_{M_{t_k}} |x^\perp| |H| = \lim_{k \rightarrow \infty} \frac{1}{t_k} \int_{\partial M_{t_k}} \frac{|x^\perp| |H|}{|\nabla r|} = 0,$$

$$(ii) \quad \lim_{k \rightarrow \infty} t_k \int_{\partial M_{t_k}} \frac{|H|^2}{|\nabla r|} = 0, \quad \lim_{k \rightarrow \infty} t_k \int_{\partial M_{t_k}} \frac{|A|^2}{|\nabla r|} = 0.$$

*Proof.* Let  $u_1(t)$ ,  $u_2(t)$  be as in Lemma 3.1. Since  $u_1(t) + u_2(t) + \int_{M_t} |H|^2 + \int_{M_t} |A|^2$  is bounded, arguing as in the proof of the claim in Lemma 3.1, we know that there is a sequence  $\{t_k\}$  diverging to infinity such that

$$(3.6) \quad t \frac{d}{dt} \left( u_1(t) + u_2(t) + \int_{M_t} |H|^2 + \int_{M_t} |A|^2 \right) \Big|_{t=t_k} \longrightarrow 0 \text{ as } k \longrightarrow \infty.$$

Since derivative of each function in left side of (3.6) is nonnegative, we have

$$(3.7) \quad \begin{aligned} t_k u_1'(t_k) &\rightarrow 0, t_k u_2'(t_k) \rightarrow 0 \text{ and} \\ t \left( \int_{M_t} |H|^2 \right)' \Big|_{t=t_k} &\rightarrow 0, t \left( \int_{M_t} |A|^2 \right)' \Big|_{t=t_k} \rightarrow 0 \end{aligned}$$

as  $k \rightarrow \infty$ . Combining (3.5) and (3.7), we get

$$(3.8) \quad \begin{aligned} t \left( \frac{v(t)}{t^2} \right)' \Big|_{t=t_k} &\rightarrow 0, t \left( \frac{\int_{M_t} |x^\perp| |H|}{t^2} \right)' \Big|_{t=t_k} \rightarrow 0 \text{ and} \\ t \left( \int_{M_t} |H|^2 \right)' \Big|_{t=t_k} &\rightarrow 0, t \left( \int_{M_t} |A|^2 \right)' \Big|_{t=t_k} \rightarrow 0 \end{aligned}$$

as  $k \rightarrow \infty$ . So we obtain

(3.9)

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{v'(t_k)}{t_k} &= \lim_{k \rightarrow \infty} \frac{2v(t_k)}{t_k^2}, \quad \lim_{k \rightarrow \infty} \frac{1}{t_k} \left( \int_{M_{t_k}} |x^\perp| |H| \right)' = \lim_{k \rightarrow \infty} \frac{2}{t_k^2} \int_{M_{t_k}} |x^\perp| |H| \quad \text{and} \\ \lim_{k \rightarrow \infty} t_k \int_{\partial M_{t_k}} \frac{|H|^2}{|\nabla r|} &= 0, \quad \lim_{k \rightarrow \infty} t_k \int_{\partial M_{t_k}} \frac{|A|^2}{|\nabla r|} = 0, \end{aligned}$$

where we use the fact that  $\lim_{t \rightarrow \infty} \frac{v(t)}{t^2}$  and  $\lim_{t \rightarrow \infty} \frac{\int_{M_t} |x^\perp| |H|}{t^2}$  exist by Lemma 3.1, this proves (ii).

By co-area formula, when  $k \rightarrow \infty$ ,

$$\begin{aligned} \left. \frac{1}{t} \frac{d}{dt} \left( \int_{M_t} |x^\perp| |H| \right) \right|_{t=t_k} &= \frac{1}{t_k} \int_{\partial M_{t_k}} \frac{|x^\perp| |H|}{|\nabla r|} \\ &\leq \int_{\partial M_{t_k}} \frac{|H|}{|\nabla r|} \\ (3.10) \quad &\leq \sqrt{v'(t_k) \int_{\partial M_{t_k}} \frac{|H|^2}{|\nabla r|}} \\ &= \sqrt{\frac{v'(t_k)}{t_k} t_k \int_{\partial M_{t_k}} \frac{|H|^2}{|\nabla r|}} \\ &\rightarrow 0. \end{aligned}$$

Combining (3.9) and (3.10), we have

$$(3.11) \quad \lim_{k \rightarrow \infty} \frac{2}{t_k^2} \int_{M_{t_k}} |x^\perp| |H| = \lim_{k \rightarrow \infty} \frac{1}{t_k} \int_{\partial M_{t_k}} \frac{|x^\perp| |H|}{|\nabla r|} = 0.$$

Then (i) follows from (3.9) and (3.11).  $\square$

**Proof of Theorem 1.1** By Proposition 2.3, we have

$$\begin{aligned} \left| \int_{\partial M_t} \left\langle A \left( \frac{\nabla r}{|\nabla r|}, \frac{\nabla r}{|\nabla r|} \right), \frac{\nabla^\perp r}{|\nabla r|} \right\rangle \right| &\leq \int_{\partial M_t} |A| \frac{|\nabla^\perp r|}{|\nabla r|} \\ (3.12) \quad &\leq \int_{\partial M_t} \left( \frac{t |A|^2}{2 |\nabla r|} + \frac{|\nabla^\perp r|^2}{2t |\nabla r|} \right) \\ &= \frac{t}{2} \int_{\partial M_t} \frac{|A|^2}{|\nabla r|} + \frac{tv' - (2v(t) + \int_{M_t} \langle x^\perp, H \rangle)}{2t^2}, \end{aligned}$$

then Lemma 3.2 implies

$$(3.13) \quad \lim_{k \rightarrow \infty} \left| \int_{\partial M_{t_k}} \left\langle A \left( \frac{\nabla r}{|\nabla r|}, \frac{\nabla r}{|\nabla r|} \right), \frac{\nabla^\perp r}{|\nabla r|} \right\rangle \right| = 0.$$



Taking  $t = t_k$  in Proposition 2.2 and letting  $k \rightarrow \infty$ , together with (3.13) and Lemma 3.2, we get

$$(3.14) \quad 2\pi \lim_{k \rightarrow \infty} \chi(M_{t_k}) - \int_M K = \lim_{k \rightarrow \infty} \frac{2v(t_k)}{t_k^2},$$

which implies

$$(3.15) \quad \lim_{t \rightarrow \infty} \frac{2v(t)}{t^2} \leq 2\pi\chi(M) - \int_M K.$$

Since the extrinsic distance is smaller than intrinsic distance, we clearly have

$$(3.16) \quad \lim_{t \rightarrow \infty} \frac{v(t)}{t^2} \geq \lim_{t \rightarrow \infty} \frac{D(t)}{t^2},$$

where  $D(t)$  is the area of geodesic balls of radius  $t$  at a fixed point.

Combining (1.4), (3.15) and (3.16), we conclude that

$$(3.17) \quad \lim_{t \rightarrow \infty} \frac{2v(t)}{t^2} = 2\pi\chi(M) - \int_M K.$$

Furthermore, by the main theorem of White [9], we know that  $\frac{1}{2\pi} \int_M K$  is an integer, so is  $\lim_{t \rightarrow \infty} \frac{v(t)}{\pi t^2}$ , and this limit must be positive by Corollary 2.5. This completes the proof of Theorem 1.1.

**Corollary 3.3** *Let  $M$  be a complete properly immersed noncompact oriented surface in  $\mathbb{R}^n$  with  $\int_M |A|^2 < 4\pi$ , then  $\chi(M) = 1$ .*

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